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Uniform Approximation by Nevai Operators

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The author establishes new direct and converse results for the weighted and unweighted uniform approximation by some rational operators of Nevai type.

Key Words: Nevai operator; GSJ weight; weighted approximation; converse theorems; saturation.

1. INTRODUCTION

Let u be a generalized smooth Jacobi weight function (we write $u \in \mathrm{GSJ}$) defined by

$$u(x) = \psi(x)(1-x)^{\gamma} \prod_{k=1}^{q} |x-t_k|^{\gamma_k}(1+x)^{\delta}, \qquad x \in (-1, 1), \tag{1}$$

where $-1 < t_1 < \ldots < t_q < 1$, $\gamma, \delta, \gamma_k > -1$, $k = 1, 2, \ldots, q$ and $0 < \psi \in \mathrm{Lip}_M\, \lambda$.

Further, let $\{p_n(u)\}_{n=0}^{\infty}$ be the corresponding system of orthonormal polynomials associated with the weight function u and denote by $x_{n,k} = x_k$, $k = 1, \ldots, n$, the zeros of $p_n(u)$ in natural order.

Then let $\lambda_n(x)$ be the nth Christoffel function corresponding to the weight u defined by

$$\lambda_n(x) = \lambda_n(u; x) = \left[\sum_{k=1}^{n} p_k^2(u; x) \right]^{-1} = \left[\sum_{k=1}^{n} \frac{\ell_{n,k}^2(x)}{\lambda_{n,k}} \right]^{-1}, \tag{2}$$

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where $\ell_{n,k}(x) = p_n(x)/p_n'(x_k)(x-x_k)$, $k=1, ..., n$, are the fundamental Lagrange polynomials and $\lambda_{n,k} = \lambda_n(u; x_k) = \lambda_k$, $k=1, ..., n$, are the corresponding Cotes numbers.

Then for every function f defined in $[-1, 1]$ consider the Nevai operator N_n given by

$$N_n(f; x) = \frac{\sum_{k=1}^{n} \frac{|\ell_{n,k}(x)|^s}{\lambda_{n,k}^{s/2}} f(x_k)}{\sum_{k=1}^{n} \frac{|\ell_{n,k}(x)|^s}{\lambda_{n,k}^{s/2}}}, \qquad x \in [-1, 1], s \geqslant 2. \tag{3}$$

From the definition (3), it follows that N_n is a positive operator interpolating f at the nodes x_k, $k=1, ..., n$, it preserves constant functions and, if s is an even integer, $N_n(f)$ is a rational function.

In the particular case $s=2$, this operator coincides with the operator F_n introduced and studied by Nevai in [12]. When $s=2$, Criscuolo *et al.* in [3] obtained pointwise error estimates for N_n, involving the usual modulus of continuity of f. Some weak asymptotic relations were also given in [3].

In [5] Della Vecchia and Mastroianni introduced a modification of N_n operator and they proved pointwise simultaneous approximation error estimates of Gopengauz-Teliakovskii type. We also remark that N_n belongs to a more general class of linear, positive, rational interpolatory operators introduced and studied by Criscuolo and Mastroianni in [2] (see also [5]). In particular in [2] a uniform convergence result of Korovkin type for N_n was established. An expression of N_n in terms of H_n, with H_n the Hermite–Fejér interpolating polynomial operator, was also showed in [2]. Moreover N_n is related to Shepard operator S_n (see (34)).

Operators N_n are of interest in applications because they can be used in approximating Christoffel functions corresponding to non-classical weight functions.

In this paper we want to investigate the more general weighted approximation case, when the function f may be unbounded at ± 1.

First we show that, similarly as for polynomials, for the operators N_n the weighted convergence with Pollaczek type weights is not guaranteed in general (Proposition 2.1). Therefore here we consider weights vanishing algebraically at ± 1, i.e. functions having an algebraic singularity at ± 1. For such functions we give weighted uniform approximation estimates by N_n involving a suitable modulus of smoothness. We also establish converse results (Theorem 2.1). Useful tools for our results are new weighted Markov–Bernstein inequalities for N_n (Lemmas 3.2–3.3). We also show that our results are sharp in some sense (see remarks to Theorem 2.1).

In the particular case of the unweighted approximation, i.e., if $f \in C([-1, 1])$, we obtain more precise direct and converse results.

Finally the difficult problem of saturation of N_n for $s \geqslant 2$ is investigated, and when $s > 2$, it is solved (Theorems 2.2 and 2.3).

2. MAIN RESULTS

Letting

$$w(x) = (1-x^2)^\alpha, \qquad \alpha > 0,\ x \in [-1, 1], \tag{4}$$

we consider functions f locally continuous on $(-1, 1)$ ($f \in C_{\mathrm{Loc}}((-1, 1))$) such that

$$\lim_{x \to \pm 1} w(x)\, f(x) = 0. \tag{5}$$

Here we want to study the weighted uniform convergence of $N_n(f)$ to f, with N_n defined by (3) and w given by (4), i.e., the convergence behaviour of $w(x)|f(x) - N_n(f; x)|$, for $|x| \leqslant 1$. First we remark that we have to consider weights of type (4) for the weighted approximation by N_n, since for Pollaczek type weights the convergence is not guaranteed in general (cf. [6] for analogous behaviour of Shepard operator).

Indeed, putting $\|wf\|_{[a,b]} = \sup_{x \in [a,b]} w(x)|f(x)|$ and $\|wf\| = \sup_{|x| \leqslant 1} w(x)|f(x)|$, we have

PROPOSITION 2.1. *Let* $u(x) = (1-x^2)^\gamma$, $\gamma > -1$ *(i.e. u is given by* (1) *with* $\gamma = \delta$ *and* $\gamma_1 = \gamma_2 = \cdots = \gamma_q = 0$*). Moreover put* $W(x) = \exp(-1/(1-x^2))$ *and* $f(x) = \exp((1-x^2)^{-1/2})$*. Then*

$$\limsup_n \|W N_n(f)\| = +\infty. \tag{6}$$

Now let

$$\omega^\varphi(f; t)_w = \sup_{0 \leqslant h \leqslant t} \|w \Delta_{h\varphi} f\|_{[-1+2h^2,\, 1-2h^2]} + \sup_{0 < h \leqslant 2t^2} \|w \overrightarrow{\Delta}_h f\|_{[-1,\, -1+2t^2]} + \sup_{0 < h \leqslant 2t^2} \|w \overleftarrow{\Delta}_h f\|_{[1-2t^2,\, 1]},$$

$$\Delta_{h\varphi} f(x) = f\left(x + h\frac{\varphi(x)}{2}\right) - f\left(x - h\frac{\varphi(x)}{2}\right),$$

$$\overrightarrow{\Delta}_h f(x) = f(x+h) - f(x),$$

$$\overleftarrow{\Delta}_h f(x) = f(x) - f(x-h),$$

be the weighted modulus of smoothness of first order of f with step function $\varphi(x) = \sqrt{1-x^2}$ and w given by (4) (cf. [8, formula (8.2.10), p. 97]).

In the following C denotes a positive constant which may assume different values in different formulas. Moreover let $\nu \sim \mu$, for ν and μ two quantities depending on some parameters, if $|\nu/\mu|^{\pm 1} \leqslant C$, with C independent of the parameters. Then we give the following direct and converse result.

THEOREM 2.1. *Let $s \geqslant 2\alpha+2$. If f satisfies condition (5), then*

$$\|w[f-N_n(f)]\| \leqslant C\omega^{\varphi}\left(f; \frac{1}{n}\right)_w \tag{7}$$

and

$$\omega^{\varphi}\left(f; \frac{1}{n}\right)_w \sim \|w[f-N_n(f)]\| + \frac{1}{n}\|w\varphi N_n'(f)\|. \tag{8}$$

In addition

$$\|w[f-N_n(f)]\| = O(n^{-\beta}) \Leftrightarrow \omega^{\varphi}(f;t)_w = O(t^{\beta}), \qquad 0<\beta<1. \tag{9}$$

Remark. From (7) we deduce the weighted uniform convergence of the operator N_n, if $s \geqslant 2\alpha+2$. As expected, our error estimates are strongly affected by the mesh distribution (see the presence of the function φ on the right-hand side of (7)).

We remark that such results can also be obtained by polynomial operators (cf. [8]) (which however are not positive), while classical positive operators of Bernstein-type give a poorer rate of convergence (cf. [8]) and do not interpolate.

From (8), by (7) we deduce (see formula (45))

$$\|w\varphi N_n'(f)\| \leqslant Cn\omega^{\varphi}\left(f; \frac{1}{n}\right)_w. \tag{10}$$

We remark that an analogous estimate holds true for the best weighted polynomial approximation to f (see [8]).

From (8), since [8]

$$\omega^{\varphi}\left(f; \frac{1}{n}\right)_w \sim K^{\varphi}\left(f; \frac{1}{n}\right)_w \tag{11}$$

with $K^{\varphi}(f)_w$ the weighted K-functional, it follows that

$$\inf_{\substack{h \in C_{\mathrm{Loc}}((-1,1)) \\ \|wh\| < +\infty \\ \|w\varphi h'\| < +\infty}} \left\{\|w[f-h]\| + \frac{1}{n}\|w\varphi h'\|\right\} \sim \|w[f-N_n(f)]\| + \frac{1}{n}\|w\varphi N_n'(f)\|, \tag{12}$$

in other words the infimum at the left-hand side in (12) is essentially realized by $N_n(f)$.

Moreover, (7) cannot be improved because of (9). In a sense, equivalence relation (9) characterizes the class of functions satisfying (5) and having a given behaviour near ± 1 by the order of approximation by N_n operator.

When the function f is continuous on the whole interval $[-1, 1]$, we can give more precise direct and converse results, solving the saturation problem of N_n, for $s > 2$. Indeed

THEOREM 2.2. *Let $s > 2$ and $f \in C([-1, 1])$. Then*

$$\|f - N_n(f)\| \leqslant C\omega^{\varphi}\left(f; \frac{1}{n}\right) \tag{13}$$

and

$$\omega^{\varphi}\left(f; \frac{1}{n}\right) \sim \|f - N_n(f)\| + \frac{1}{n}\|\varphi N_n'(f)\|. \tag{14}$$

In addition if $f \neq$ constant

$$\limsup_{n \to +\infty} \frac{\|N_n(f) - f\|}{\omega^{\varphi}\left(f; \frac{1}{n}\right)} \sim 1, \tag{15}$$

where the sign $\sim$ does not depend on f.

Moreover

$$\|N_n(f) - f\| = o\left(\frac{1}{n}\right) \Leftrightarrow f \text{ is a constant}, \tag{16}$$

$$\|N_n(f) - f\| = O\left(\frac{1}{n}\right) \Leftrightarrow \omega^{\varphi}(f; t) \leqslant Ct. \tag{17}$$

Remarks. First note that direct estimate (13) cannot be improved because of (15).

Estimation (15) is a counterpart of (13) and has a character similar to the result of Totik [16, (1.2)]

$$\|B_n(f) - f\|_{[0,1]} \sim \omega_{\psi}^2\left(f; \frac{1}{\sqrt{n}}\right), \tag{18}$$

with $B_n(f)$ the nth Bernstein polynomial, $f \in C([0, 1])$, $\|\cdot\|_{[0,1]}$ the usual supremum norm on $[0, 1]$ and ω_{ψ}^2 the second modulus of smoothness of

Ditzian and Totik with $\psi(x)=\sqrt{x(1-x)}$. However, due to the interpolatory character of N_n, we cannot get the estimation (15) with"lim" (instead of "lim sup") as a consequence of a result stated by Della Vecchia *et al.* in [7, p. 77] (cf. also [17, Theorem 2.1, p. 310]).

Estimation (15) combined with the equivalence relation (see, e.g., [8]) $\omega^{\varphi}(f;t)\sim K^{\varphi}(f;t)$, with $K^{\varphi}(f)$ the K-functional with step-function φ, can serve as a characterization of such K-functionals.

Finally we remark that (16)–(17) handle the saturation problem for N_n with $s>2$.

We remark that the assumption $s>2$ in Theorem 2.2 is essential: indeed the case $s=2$ presents additional difficulties because we do not have strong localization theorems like for the case $s>2$. Some contributions to the saturation problem of $N_n(f)$ if $s=2$ were given by Criscuolo *et al.* in [3]. In particular (cf. [3])

$$\omega(f;t)\leqslant Ct^{\beta},\ 0<\beta<1 \Rightarrow \|f-N_n(f)\|=O(n^{-\beta}), \tag{19}$$

$$\omega(f;t)\leqslant Ct \Rightarrow \|f-N_n(f)\|=O\left(\frac{\log n}{n}\right), \tag{20}$$

with $\omega(f)$ the usual modulus of continuity of f.

Here we have

THEOREM 2.3. *Let $s=2$. Then the following implications hold*

$$\omega^{\varphi}(f;t)\leqslant Ct^{\beta},\ 0<\beta<1 \Rightarrow \|f-N_n(f)\|\leqslant Cn^{-\beta}, \tag{21}$$

$$\omega^{\varphi}(f;t)\leqslant Ct \Rightarrow \|f-N_n(f)\|\leqslant C\,\frac{\log n}{n}, \tag{22}$$

$$\|f-N_n(f)\|=O(n^{-1}) \Rightarrow \omega^{\varphi}(f;t)\leqslant Ct^{\beta}, \qquad \forall\beta\in(0,1). \tag{23}$$

Remark. The estimate (22) seems exact, in the sense that, following [11, pp. 11–14] we can find a function f such that $\omega^{\varphi}(f;t)\leqslant Ct$ implies that

$$\|f-N_n(f)\|\geqslant C\,\frac{\log n}{n}.$$

3. PROOFS OF THE MAIN RESULTS

First we give some preliminary results which will be useful in the sequel.

We recall that if $x_o = -1$, $x_{n+1} = 1$ and x_k, $k = 1, \ldots, n$, denote the zeros of $p_n(u) = p_n$ with u given by (1), then $x_k = \cos\theta_k$, $\theta_k \in [0, \pi]$, $k = 0, 1, \ldots, n+1$, and

$$\theta_k - \theta_{k+1} \sim \frac{1}{n}, \qquad k = 0, \ldots, n \tag{24}$$

(see, e.g., [12, 13]).

Moreover [13, p. 673]

$$\frac{1}{|p_n'(x_k)|} \sim \lambda_k |p_{n-1}(x_k)|. \tag{25}$$

In addition [13, p. 673] for $k = 1, \ldots, n$

$$|p_{n-1}(x_k)| \sim (1-x_k)^{-\gamma/2+1/4} (1+x_k)^{-\delta/2+1/4} \prod_{j=1}^{q} (|t_j - x_k| + n^{-1})^{-\gamma_j/2} \tag{26}$$

and [13]

$$\lambda_k \sim \frac{1}{n} (1-x_k)^{\gamma+1/2} (1+x_k)^{\delta+1/2} \prod_{j=1}^{q} (|t_j - x_k| + n^{-1})^{\gamma_j}, \qquad k = 1, \ldots, n. \tag{27}$$

Consequently

$$\frac{|\ell_{n,k}(x)|}{\lambda_{n,k}^{1/2}} \sim \frac{|p_n(x)|(1-x_k^2)^{1/2}}{\sqrt{n}\,|x - x_k|}, \qquad k = 1, \ldots, n. \tag{28}$$

We also recall that if x_j denotes the closest zero to x, then [13, p. 673]

$$\frac{1-x_k^2}{n^2(x-x_k)^2} \leqslant \frac{1}{(k-j)^2}, \qquad k \neq j, \tag{29}$$

$$|x - x_k| \sim \frac{|j-k|}{n} \sqrt{1-x^2} + \frac{(j-k)^2}{n^2}, \qquad k \neq j, \tag{30}$$

and

$$|x - x_j| \leqslant C \left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2} \right). \tag{31}$$

Moreover [13, p. 673]

$$1 - x_n \sim 1 + x_1 \sim \frac{1}{n^2} \tag{32}$$

and

$$\max_{|x|\leqslant 1} n\lambda_n(x)\, p_n^2(x) \leqslant C. \tag{33}$$

Consequently by (28)–(32)

$$N_n(|f|;x) \sim \frac{\sum_{k=1}^{n} \frac{(1-x_k^2)^{s/2}}{|x-x_k|^s} |f(x_k)|}{\sum_{k=1}^{n} \frac{(1-x_k^2)^{s/2}}{|x-x_k|^s}}$$
$$\leqslant \frac{C|x-x_j|^s}{(1-x_j^2)^{s/2}} \left[\sum_{k\neq j} \frac{|f(x_k)|}{|j-k|^s} n^s \right] + |f(x_j)|$$
$$\leqslant C \left[\sum_{k\neq j} \frac{|f(x_k)|}{|j-k|^s} + |f(x_j)| \right]. \tag{34}$$

This shows that N_n is related to Shepard operator S_n in some sense (see [2]). Since the weighted behaviour of Shepard-type operators is unknown in general (it is the subject of a future paper) and our demonstration techniques are based on direct estimates for N_n (see, e.g., the proofs of Lemmas 3.2 and 3.3 and Theorem 2.2), here we had to work directly on N_n operators.

First we prove Proposition 2.1.

Proof. Let n be even. From (28) we obtain

$$w(0)|N_n(f;0)| = \frac{\exp(-1) \sum_{k=1}^{n} \frac{|\ell_{n,k}(0)|^s}{\lambda_{n,k}^{s/2}} \exp\left(\frac{1}{\sqrt{1-x_k^2}}\right)}{\sum_{k=1}^{n} \frac{|\ell_{n,k}(0)|^{s/2}}{\lambda_{n,k}^{s}}}$$
$$\sim \frac{\sum_{k=1}^{n} \frac{(1-x_k^2)^{s/2} \exp\left(\frac{1}{\sqrt{1-x_k^2}}\right)}{|x_k|^s}}{\sum_{k=1}^{n} \frac{(1-x_k^2)^{s/2}}{|x_k|^s}}$$
$$\geqslant \frac{C(1-x_n^2)^{s/2} \exp\left(\frac{1}{\sqrt{1-x_n^2}}\right)}{\sum_{k=1}^{n} \frac{(1-x_k^2)^{s/2}}{|x_k|}}. \tag{35}$$

Since $|x_k| \geqslant \frac{C}{n}$, $k = \frac{n}{2}, \frac{n}{2}+1$, by (29) we deduce

$$\sum_{k=1}^{n} \frac{(1-x_k^2)^{s/2}}{|x_k|^s} \leqslant C\left\{n^s \sum_{k \neq \frac{n}{2}, \frac{n}{2}+1} \frac{1}{|k-n/2|^s} + n^s\right\} \leqslant Cn^s. \tag{36}$$

Therefore from (32), (35), and (36),

$$w(0)|N_n(f; 0)| \geqslant \frac{C \exp(Cn)(1-x_n^2)^{s/2}}{n^s} \geqslant C \frac{\exp(Cn)}{n^{2s}}$$

which is unbounded when $n \to +\infty$. The assertion follows. ∎

The following lemma will be useful in the sequel. It establishes the boundedness of the operator N_n in the weighted norm.

LEMMA 3.1. *Let $s \geqslant 2\alpha + 1$. Then for every function f defined on $[-1, 1]$ we have*

$$\|wN_n(f)\| \leqslant C\|wf\| \left\|wN_n\left(\frac{1}{w}\right)\right\| \leqslant C\|wf\| \tag{37}$$

with C a positive constant independent of f and n.

Proof. Because of the interpolatory property of N_n at x_k, $k = 1, \ldots, n$, we may assume $x \neq x_k$, $k = 1, \ldots, n$. Assume $x \geqslant 0$. Similarly we work if $x < 0$.

We distinguish three cases.

Case 1. $x > x_k > 0$.

Then $w(x) \sim (1-x)^\alpha < (1-x_k)^\alpha \sim w(x_k)$, therefore

$$w(x) \frac{\sum_{0<x_k<x} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{s/2}} |f(x_k)|}{\sum_{k=1}^{n} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{s/2}}} \leqslant \|wf\| w(x) \frac{\sum_{0<x_k<x} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{s/2}} \frac{1}{w(x_k)}}{\sum_{k=1}^{n} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{s/2}}} \leqslant C\|wf\|.$$

Case 2. $0 < x < x_k$.

Here by (34)

$$T := w(x) \frac{\sum_{x_k > x} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{s/2}} |f(x_k)|}{\sum_{k=1}^{n} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{s/2}}}$$

$$\leqslant \|wf\| w(x) \frac{\sum_{x_k > x} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{s/2}} \frac{1}{w(x_k)}}{\sum_{k=1}^{n} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{s/2}}}$$

$$= C\|wf\|(1-x)^\alpha \left\{ \sum_{k=j+1}^{(n+j)/2} + \sum_{k=(n+j)/2+1}^{n} \right\} \frac{1}{|k-j|^s} \frac{1}{(1-x_k)^\alpha}$$

$$:= \|wf\|(S_1 + S_2), \tag{38}$$

where again x_j denotes the closest knot to x.

Now by (30)

$$S_1 \leqslant C(1-x)^\alpha \sum_{k=j+1}^{(n+j)/2} \frac{1}{(k-j)^s} \frac{n^{2\alpha}}{(n-k)^{2\alpha}}$$

$$\leqslant C(1-x)^\alpha \frac{n^{2\alpha}}{(n-(n+j)/2)^{2\alpha}} \sum_{k=j+1}^{(n+j)/2} \frac{1}{(k-j)^s}$$

$$\leqslant C \frac{(1-x)^\alpha}{(1-x_j)^\alpha} \leqslant C. \tag{39}$$

On the other hand by (32) and (30)

$$S_2 \leqslant C(1-x)^\alpha \sum_{k=(n+j)/2+1}^{n} \frac{1}{(k-j)^s} \frac{1}{(1-x_k)^\alpha}$$

$$\leqslant C(1-x)^\alpha \frac{n^{2\alpha}}{(n-j+1/2)^{s-1}}$$

$$\leqslant C \tag{40}$$

if $s \geqslant 2\alpha + 1$.

Hence from (38), by (39) and (40)

$$T \leqslant C\|wf\|$$

if $s \geqslant 2\alpha + 1$.

Case 3. $x_k < 0$.

Here by (30) and (34)

$$W := w(x) \frac{\sum_{x_k<0} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{s/2}} |f(x_k)|}{\sum_{k=1}^{n} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{s/2}}}$$

$$\leqslant \|wf\| w(x) \frac{\sum_{x_k<0} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{s/2}} \frac{1}{w(x_k)}}{\sum_{k=1}^{n} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{s/2}}}$$

$$\leqslant C\|wf\|(1-x)^{\alpha} n^{2\alpha} \left\{ \sum_{k=1}^{j/2} + \sum_{k=j/2+1}^{n/2} \right\} \frac{1}{k^{2\alpha}(j-k)^s}$$

$$\leqslant C\|wf\|(1-x)^{\alpha} n^{2\alpha} \left\{ \frac{1}{j^s} j + \frac{1}{(j/2+1)^{2\alpha}} \frac{1}{(j-n/2)^{s-1}} \right\}$$

$$\leqslant C\|wf\|$$

if $s-1 \geqslant 2\alpha$.

Hence the assertion is proved. ▮

Note that Lemma 3.1 does not need the assumption (5).

The following lemmas are useful to prove Theorems 2.1 and 2.3. In particular they are interesting in themselves because they establish some weighted Markov–Bernstein type inequalities for the operator N_n.

LEMMA 3.2. *If* $s \geqslant 2\alpha + 1$, *then*

$$\|w\varphi N_n'(f)\| \leqslant Cn\|wf\|,$$

with $\varphi = \sqrt{1-x^2}$ *and C independent of f and n.*

Proof. Since $N_n'(f; x_k) = 0$, $k = 1, ..., n$, we assume $x \neq x_k$, $k = 1, ..., n$. From (3)

$$N_n(f; x) = \frac{\sum_{k=1}^{n} \frac{f(x_k)}{|x-x_k|^s |p_n'(x_k)|^s \lambda_k^{s/2}}}{\sum_{k=1}^{n} \frac{1}{|x-x_k|^s |p_n'(x_k)|^s \lambda_k^{s/2}}}. \tag{41}$$

Hence

$$N_n'(f;x)=\frac{-s\sum_{k=1}^{n}\frac{f(x_k)}{|x-x_k|^{s+1}|p_n'(x_k)|^s\lambda_k^{s/2}}\sum_{k=1}^{n}\frac{1}{|x-x_k|^s|p_n'(x_k)|^s\lambda_k^{s/2}}}{\left[\sum_{k=1}^{n}\frac{1}{|x-x_k|^s|p_n'(x_k)|^s\lambda_k^{s/2}}\right]^2}$$

$$+\frac{s\sum_{k=1}^{n}\frac{f(x_k)}{|x-x_k|^{s}|p_n'(x_k)|^s\lambda_k^{s/2}}\sum_{k=1}^{n}\frac{1}{|x-x_k|^{s+1}|p_n'(x_k)|^s\lambda_k^{s/2}}}{\left[\sum_{k=1}^{n}\frac{1}{|x-x_k|^s|p_n'(x_k)|^s\lambda_k^{s/2}}\right]^2}$$

and

$$|N_n'(f;x)|\leqslant\frac{C}{\left[\sum_{k=1}^{n}\frac{1}{|x-x_k|^s|p_n'(x_k)|^s\lambda_k^{s/2}}\right]^2}$$

$$\left\{\sum_{k\neq j}\frac{|f(x_k)|}{|x-x_k|^{s+1}|p_n'(x_k)|^s\lambda_k^{s/2}}\sum_{k\neq j}\frac{1}{|x-x_k|^{s}|p_n'(x_k)|^s\lambda_k^{s/2}}\right.$$

$$+\sum_{k\neq j}\frac{|f(x_k)|}{|x-x_k|^{s}|p_n'(x_k)|^s\lambda_k^{s/2}}\sum_{k\neq j}\frac{1}{|x-x_k|^{s+1}|p_n'(x_k)|^s\lambda_k^{s/2}}$$

$$+\frac{|f(x_j)|}{|x-x_j|^{s+1}|p_n'(x_j)|^s\lambda_j^{s/2}}\sum_{k\neq j}\frac{1}{|x-x_k|^{s}|p_n'(x_k)|^s\lambda_k^{s/2}}$$

$$+\sum_{k\neq j}\frac{|f(x_k)|}{|x-x_k|^{s+1}|p_n'(x_k)|^s\lambda_k^{s/2}}\frac{1}{|x-x_j|^{s}|p_n'(x_j)|^s\lambda_j^{s/2}}$$

$$+\frac{|f(x_j)|}{|x-x_j|^{s}|p_n'(x_j)|^s\lambda_j^{s/2}}\sum_{k\neq j}\frac{1}{|x-x_k|^{s+1}|p_n'(x_k)|^s\lambda_k^{s/2}}$$

$$\left.+\sum_{k\neq j}\frac{|f(x_k)|}{|x-x_k|^{s}|p_n'(x_k)|^s\lambda_k^{s/2}}\frac{1}{|x-x_j|^{s+1}|p_n'(x_j)|^s\lambda_j^{s/2}}\right\}$$

$$:=S_1+S_2+S_3+S_4+S_5+S_6. \tag{42}$$

First we note that by (25)–(27)

$$\frac{1}{\left[\sum_{k=1}^{n}\frac{1}{|x-x_k|^s|p_n'(x_k)|^s\lambda_k^{s/2}}\right]^2}\leqslant\frac{C|x-x_j|^{2s}}{\lambda_j^s|p_{n-1}(x_j)|^{2s}}. \tag{43}$$

First we estimate $\|w\varphi S_5\|$.

From (25)–(27), (29)–(31), and (43)

$$
\begin{aligned}
w(x)\,\varphi(x)|S_5| &\leqslant C\,\frac{w(x)\,\varphi(x)|x-x_j|^{2s}}{\lambda_j^s|p_{n-1}(x_j)|^{2s}}\,\lambda_j^{s/2}\,\frac{|p_{n-1}(x_j)|^s}{|x-x_j|^s}\,|f(x_j)| \\
&\quad\times\sum_{k\neq j}\frac{\lambda_k^{s/2}|p_{n-1}(x_k)|^{s/2}}{|x-x_k|^{s+1}} \\
&\leqslant C\|wf\|\,\frac{|x-x_j|^s}{n^{-s/2}(1-x_j^2)^{s/2}}\,\varphi(x)\sum_{k\neq j}\frac{(1-x_k^2)^{s/2}}{n^{s/2}|x-x_k|^{s+1}} \\
&\leqslant Cn\|wf\|\,\frac{(\sqrt{1-x^2}+1/n)^s}{(1-x_j)^{s/2}}\sum_{k\neq j}\frac{1}{n^s}\,\frac{(1-x_k^2)^{s/2}}{|x-x_k|^s} \\
&\leqslant Cn\|wf\|.
\end{aligned}
$$

Similarly by Lemma 3.1 we can prove that

$$\|w\varphi S_i\|\leqslant Cn\|wf\|,\ i=1,2,3,4,6.$$

Hence the assertion follows. ∎

Lemma 3.3. *Let $s\geqslant 2\alpha+2$. Then*

$$\|w\varphi N_n'(f)\|\leqslant C\|w\varphi f'\|.$$

Proof. We assume $x\neq x_k$, $k=0,\ldots,n$. It results that

$$N_n(f;x)=\sum_{k=1}^{n}A_k(x)\,f(x_k)=f(x)+\sum_{k=1}^{n}A_k(x)[f(x_k)-f(x)],\tag{44}$$

with

$$A_k(x)=\frac{|\ell_{n,k}(x)|^s/\lambda_k^{s/2}}{\sum_{k=1}^{n}|\ell_{n,k}(x)|^s/\lambda_k^{s/2}}.$$

Since $\sum_{k=1}^{n}A_k'(x)=0$, from (44) it follows that

$$N_n'(f;x)=\sum_{k=1}^{n}A_k'(x)[f(x_k)-f(x)].$$

Hence by the mean value theorem

$$
\begin{aligned}
w(x)\,\varphi(x)|N_n'(f;x)| &\leqslant w(x)\,\varphi(x)\sum_{k=1}^{n}|A_k'(x)||g(\theta_k)-g(\theta)| \\
&\leqslant Cw(x)\,\varphi(x)\sum_{k=1}^{n}\frac{|A_k'(x)|}{w(\xi_k)}\|g'w\|_{[0,\pi]}|\theta-\theta_k| \\
&\leqslant Cw(x)\,\varphi(x)\|f'w\varphi\|\sum_{k=1}^{n}\frac{|A_k'(x)||\theta-\theta_k|}{w(\xi_k)}
\end{aligned}
$$

where $g(\theta)=f(\cos\theta)$, $\bar{w}(\theta)=w(\cos\theta)$, $\|g'\bar{w}\|_{[0,\pi]}$ is the usual supremum norm on $[0,\pi]$ of $g'\bar{w}$ and ξ_k lies between x and x_k.

Now if $w(\xi_k)>w(x)$, then we need to estimate $\sum_{k=1}^{n}|A_k'(x)||\theta-\theta_k|\varphi(x)$. From (42) and (24), if x_j denotes again the closest zero to x, we have

$$
\begin{aligned}
\varphi(x)\sum_{k=1}^{n}|A_k'(x)||\theta-\theta_k| \leqslant{}& \frac{C\varphi(x)}{n\left[\sum_{k=1}^{n}\frac{\lambda_k^{s/2}|p_{n-1}(x_k)|^s}{|x-x_k|^s}\right]^2} \\
&\left\{\sum_{k\neq j}\frac{\lambda_k^{s/2}|p_{n-1}(x_k)|^s}{|x-x_k|^{s+1}}|k-j|\sum_{k\neq j}\frac{\lambda_k^{s/2}|p_{n-1}(x_k)|^s}{|x-x_k|^s}\right. \\
&+\sum_{k\neq j}\frac{\lambda_k^{s/2}|p_{n-1}(x_k)|^s}{|x-x_k|^s}|k-j|\sum_{k\neq j}\frac{\lambda_k^{s/2}|p_{n-1}(x_k)|^s}{|x-x_k|^{s+1}} \\
&+\frac{\lambda_j^{s/2}|p_{n-1}(x_j)|^s}{|x-x_j|^{s+1}}\sum_{k\neq j}\frac{\lambda_k^{s/2}|p_{n-1}(x_k)|^s}{|x-x_k|^s} \\
&+\sum_{k\neq j}\frac{\lambda_k^{s/2}|p_{n-1}(x_k)|^s}{|x-x_k|^{s+1}}|k-j|\frac{\lambda_j^{s/2}|p_{n-1}(x_j)|^s}{|x-x_j|^s} \\
&+\frac{\lambda_j^{s/2}|p_{n-1}(x_j)|^s}{|x-x_j|^s}\sum_{k\neq j}\frac{\lambda_k^{s/2}|p_{n-1}(x_k)|^s}{|x-x_k|^{s+1}} \\
&\left.+\sum_{k\neq j}\frac{\lambda_k^{s/2}|p_{n-1}(x_k)|^s}{|x-x_k|^s}|k-j|\frac{\lambda_j^{s/2}|p_{n-1}(x_j)|^s}{|x-x_j|^{s+1}}\right\} \\
:={}& T_1+T_2+T_3+T_4+T_5+T_6.
\end{aligned}
$$

We start in estimating T_5. Indeed by (43), (25)–(27), and (29)–(31),

$$
\begin{aligned}
T_5 &\leqslant C\frac{|x-x_j|^s}{\lambda_j^{s/2}|p_{n-1}(x_j)|^s}\sum_{k\neq j}\frac{\lambda_k^{s/2}|p_{n-1}(x_k)|^s}{|x-x_k|^s} \\
&\leqslant C\frac{(\sqrt{1-x^2}+1/n)^s}{n^{s/2}(1-x_j^2)^{s/2}}\sum_{k\neq j}\frac{(1-x_k^2)^{s/2}}{n^{s/2}|x-x_k|^s}\leqslant C.
\end{aligned}
$$

Working similarly

$$T_i \leqslant C, \qquad i = 1, 2, 3, 4, 6.$$

If $w(\xi_k) < w(x)$, then we work similarly and by following the proof of Lemma 3.1 with $s-1 \geqslant 2\alpha+1$, finally we get the assertion. ∎

We remark that from Lemmas 3.2 and 3.3 we obtain for $h \in C_{\mathrm{Loc}}((-1, 1))$, $\|wh\| < +\infty$, and $\|wh'\varphi\| < +\infty$

$$\begin{aligned}\|w\varphi N_n'(f)\| &\leqslant \|w\varphi N_n'(f-h)\| + \|w\varphi N_n'(h)\| \\ &\leqslant C\{n\|w[f-h]\| + \|w\varphi h'\|\} \\ &\leqslant Cn\left\{\|w[f-h]\| + \frac{1}{n}\|w\varphi h'\|\right\}.\end{aligned}$$

By (11) we get

$$\|w\varphi N_n'(f)\| \leqslant CnK^{\varphi}\left(f; \frac{1}{n}\right)_w \leqslant Cn\omega^{\varphi}\left(f; \frac{1}{n}\right)_w. \tag{45}$$

Now we demonstrate Theorem 2.1.

Proof. Obviously we assume $x \neq x_k$, $k = 1, ..., n$. Then

$$\begin{aligned} w(x)|f(x) - N_n(f; x)| &\leqslant w(x) \frac{\displaystyle\sum_{k=1}^{n} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{s/2}} |f(x) - f(x_k)|}{\displaystyle\sum_{k=1}^{n} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{s/2}}} \\ &= w(x) \frac{\displaystyle\sum_{k=1}^{n} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{s/2}} |g(\theta) - g(\theta_k)|}{\displaystyle\sum_{k=1}^{n} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{s/2}}} \end{aligned} \tag{46}$$

with $g(\theta) = f(\cos\theta)$.

Hence if $\bar{w}(\theta) = w(\cos\theta)$ and x_j denotes again a closest knot to x, by (46) and (24) we obtain for $\|g'\bar{w}\|_{[0,\pi]} < +\infty$

$$\begin{aligned} w(x)|f(x) - N_n(f; x)| &\leqslant Cw(x) \frac{\displaystyle\sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{s/2}} \frac{\|g'\bar{w}\|_{[0,\pi]} \left|\frac{j}{n} - \frac{k}{n}\right|}{\bar{w}(\xi_k)}}{\displaystyle\sum_{k=1}^{n} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{s/2}}} \\ &\quad + C\frac{\|g'\bar{w}\|_{[0,\pi]}}{n}, \end{aligned}$$

where $\|\bar{w}g'\|_{[0,\pi]}$ denotes the usual supremum norm on $[0, \pi]$ of the bounded function $g'\bar{w}$ and ξ_k is between θ and θ_k. By (34)

$$\frac{w(x) \sum_{\bar{w}(\xi_k) > w(x)} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{s/2} \bar{w}(\xi_k)} \frac{|j-k|}{n}}{\sum_{k=1}^{n} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{s/2}}} \leqslant \frac{C}{n} \sum_{k \neq j} \frac{1}{|j-k|^{s-1}} \leqslant \frac{C}{n}.$$

Moreover

$$T := \frac{w(x) \sum_{\bar{w}(\xi_k) > w(x_k)} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{s/2} \bar{w}(\xi_k)} \frac{|j-k|}{n}}{\sum_{k=1}^{n} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{s/2}}}$$

$$\leqslant \frac{w(x) \sum_{\bar{w}(\xi_k) > w(x_k)} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{s/2} w(x_k)} \frac{|j-k|}{n}}{\sum_{k=1}^{n} \frac{|\ell_{n,k}(x)|^s}{\lambda_k^{s/2}}}$$

and working as in the proof of Lemma 3.1 we deduce for $s \geqslant 2\alpha + 2$

$$T \leqslant \frac{C}{n}.$$

Hence if $s \geqslant 2\alpha + 2$

$$\|w[f - N_n(f)]\| \leqslant C \frac{\|g'\bar{w}\|_{[0,\pi]}}{n} \leqslant C \frac{\|w\varphi f'\|}{n}, \tag{47}$$

with $\varphi(x) = \sqrt{1 - x^2}$, from which by (11)

$$\|w[f - N_n(f)]\| \leqslant C\omega^{\varphi}\left(f; \frac{1}{n}\right)_w, \qquad \forall f \in C([-1, 1]),$$

that is, (7).

Now we prove (8). By (11), (7), and (45) we obtain

$$\omega^{\varphi}\left(f; \frac{1}{n}\right)_w \leqslant CK^{\varphi}\left(f; \frac{1}{n}\right)_w \leqslant C\left\{\|w[f - N_n(f)]\| + \frac{1}{n} \|w\varphi N_n'(f)\|\right\}$$

$$\leqslant C\omega^{\varphi}\left(f; \frac{1}{n}\right)_w,$$

that is, (8).

Now we prove (9). From (7) it follows that if $\omega^{\varphi}(f;t)_w = O(t^{\beta})$, then $\|w[f-N_n(f)]\| = O(n^{-\beta})$. Now we prove the converse implication. From the definition of $K^{\varphi}(f)_w$, we obtain for $h \in C_{\mathrm{Loc}}((-1,1))$, $\|wh\| < +\infty$, $\|wh'\varphi\| < +\infty$

$$K^{\varphi}\left(f; \frac{1}{n}\right)_w \leqslant \|w[f-N_k(f)]\| + \frac{1}{n}\|w\varphi N'_k(f)\|$$

$$\leqslant \|w[f-N_k(f)]\| + \frac{1}{n}\{\|w\varphi N'_k(f-h)\| + \|w\varphi N'_k(h)\|\}.$$

Now, by using Lemmas 3.2 and 3.3 we get

$$K^{\varphi}\left(f; \frac{1}{n}\right)_w \leqslant \|w[f-N_k(f)]\| + C\frac{k}{n}\{\|w[f-h]\| + \frac{1}{k}\|w\varphi h'\|\}$$

and consequently

$$K^{\varphi}\left(f; \frac{1}{n}\right)_w \leqslant \|w[f-N_k(f)]\| + C\frac{k}{n}K^{\varphi}\left(f; \frac{1}{k}\right)_w.$$

Now if $\|w[f-N_n(f)]\| = O(n^{-\beta})$, $0<\beta<1$, then

$$K^{\varphi}\left(f; \frac{1}{n}\right)_w \leqslant Ck^{-\beta} + C\frac{k}{n}K^{\varphi}\left(f; \frac{1}{k}\right)_w$$

and from a well-known lemma by Berens and Lorentz (see, e.g., [1; 8, Lemma 9.34, p. 699]), (9) follows. ∎

Now we give the proof of Theorem 2.2.

Proof. Estimates (13) and (14) can be deduced working similarly as in the proof of (7) and (8), respectively.

Now we prove (15). From (3) the N_n operator can be written as

$$N_n(f;x) = \sum_{k=1}^{n} a_k(\theta)\, g(\theta_k) := L_n(g;\theta) \tag{48}$$

with $x = \cos\theta$, $x_k = \cos\theta_k$, $k = 1, \ldots, n$ (see 24),

$$a_k(\theta) = \frac{\dfrac{|\ell_{n,k}(\cos\theta)|^s}{\lambda_k^{s/2}}}{\displaystyle\sum_{k=1}^{n} \dfrac{|\ell_{n,k}(\cos\theta)|^s}{\lambda_k^{s/2}}} \tag{49}$$

and $g(\theta) = f(\cos\theta) \in C([0,\pi])$.

Now if we prove that

$$L_n(g;\theta)=g(\theta), \qquad \text{if} \quad g=\text{constant} \tag{50}$$

$$\sum_{|\theta-\theta_k|\geqslant d_0} |a_k(\theta)|=o\left(\frac{1}{n}\right), \qquad d_0>0 \text{ arbitrarily fixed} \tag{51}$$

$$a_j(\theta)>\frac{1}{2}, \text{ if } |\theta-\theta_j|\leqslant\frac{\delta}{n}, \qquad 0\leqslant\delta<d_1 \tag{52}$$

$$\sum_{k\neq j} |\theta-\theta_k||a_k(\theta)|\leqslant d_2\,\frac{\delta^{1+\varepsilon}}{n}, \qquad \delta \text{ is as above}, \tag{53}$$

with x_j again the closest knot to x and with certain positive fixed reals d_1, d_2 and ε, then by [9, Theorem 2.1] (see also [14; 17, Theorem 2.1])

$$\limsup_{n\to+\infty} n\,\|L_n(g)-g\|_{[0,\pi]}>KM(g), \tag{54}$$

where with $\theta, \tau\in[0,\pi]$

$$M(g):=\sup_{\theta}\left\{M(g,\theta);\, M(g,\theta):=\limsup_{\tau\to\theta}\left|\frac{g(\theta)-g(\tau)}{\theta-\tau}\right|\right\}$$

$\|\cdot\|_{[0,\pi]}$ denotes the usual supremum norm on $[0,\pi]$ and K is an absolute constant.

Now we prove properties (50)–(53).

Relation (50) holds true by definition. Now we prove (51). Since $|x-x_k|\geqslant D_0$ if $|\theta-\theta_k|\geqslant d_0$, from (49) by (25) it follows

$$\sum_{|\theta-\theta_k|>d_0} a_k(\theta)\leqslant C\,\frac{\sum\limits_{|\theta-\theta_k|>d_0}\lambda_k^{s/2}|p_{n-1}(x_k)|^s}{\sum\limits_{k=1}^{n}\dfrac{\lambda_k^{s/2}|p_{n-1}(x_k)|^s}{|\cos\theta-x_k|^s}}.$$

Then by (26), (27), and (46)

$$\sum_{|\theta-\theta_k|>d_0} a_k(\theta)\leqslant\frac{C}{n^s}\sum_{|\theta-\theta_k|>d_0}(1-x_k^2)^{s/2}\leqslant\frac{C}{n^{s-1}}=o\left(\frac{1}{n}\right),$$

that is, (51). Now we prove (52). From (25)–(30) and (2)

$$\sum_{k\neq j} a_k(\theta) \leqslant \frac{\left\{ \sum_{\substack{|x-x_k|\leqslant \varepsilon \\ k\neq j}} + \sum_{|x-x_k|>\varepsilon} \right\} \frac{|p_n(x)|^s}{|p_n'(x_k)|^s \lambda_k^{s/2} |x-x_k|^s}}{\frac{|\ell_j(x)|^s}{\lambda_j^{s/2}}}$$

$$\leqslant \frac{C|x-x_j|^s}{\lambda_j^{s/2}|p_{n-1}(x_j)|^s} \left\{ \sum_{\substack{|x-x_k|\leqslant \varepsilon \\ k\neq j}} \frac{(1-x_k^2)^{s/2}}{n^{s/2}|x-x_k|^s} + \sum_{|x-x_k|>\varepsilon} \frac{(1-x_k^2)^{s/2}}{n^{s/2}\varepsilon^s} \right\}$$

$$\leqslant \frac{C\delta^s(\varphi(x)+\delta/n)^s}{(\varphi(x)+1/n)^s} \left\{ \sum_{\substack{|x-x_k|\leqslant \varepsilon \\ k\neq j}} \frac{1}{|k-j|^s} + \frac{n}{n^s\varepsilon^s} \right\}$$

$$\leqslant C\delta^s\{1+n^{1-s}\} \leqslant \frac{1}{2},$$

if d_1 is small enough. Now by $a_k \geqslant 0$ and $\sum a_k = 1$, we get (52).

Finally we verify (53). Let $\theta < \theta_j < \theta_k$; the other cases are similar. Then

$$|\theta-\theta_k| \geqslant |\theta-\theta_j| + \frac{|\theta_j-\theta_k|}{2} \geqslant |\theta-\theta_j| \left(1 + C\,\frac{|j-k|}{\delta}\right) \geqslant \frac{C}{\delta}\,|\theta-\theta_j||k-j|.$$

Hence working as above we get

$$\sum_{k\neq j} |\theta-\theta_k| a_k(\theta) \leqslant C\,|\theta-\theta_j|^s \left\{ \sum_{\substack{|x-x_k|\leqslant \varepsilon \\ k\neq j}} \frac{1}{|\theta-\theta_k|^{s-1}} + \frac{n}{\varepsilon^s} \right\}$$

$$\leqslant C\left\{ |\theta-\theta_j|\delta^{s-1} + \frac{\delta^s}{n^{s-1}\varepsilon^s} \right\}$$

$$\leqslant C\,\frac{\delta^s}{n} + C\,\frac{\delta^s}{n^{s-1}} \leqslant C\,\frac{\delta^s}{n},$$

i.e. relation (53) has been proved.

Moreover since [10]

$$\omega^{\varphi}(f;t) \sim \omega(g;t) \tag{55}$$

with $g(\theta) = f(\cos\theta)$ and $\omega(g)$ the usual modulus of continuity of g, then (15) in Theorem 2.2 gives

$$\|f - N_n(f)\| = \|g - L_n(g)\|_{[0,\pi]} \leqslant C\omega\left(g; \frac{1}{n}\right). \tag{56}$$

From (54) and (56) we obtain (cf. [17, p. 315])

$$
\begin{aligned}
C_1 M(g) &\leqslant m \,\|L_m(g)-g\|_{[0,\pi]} \leqslant C_2 m\omega\left(g; \frac{1}{m}\right) \\
&\leqslant C_2 r_m m \frac{|g(u_m+r_m)-g(u_m)|}{r_m} \leqslant C_2 \frac{|g(u_m+r_m)-g(u_m)|}{r_m} \\
&\leqslant C_2 \sup_{\theta \neq \eta} \frac{|g(\theta)-g(\eta)|}{|\theta-\eta|} := C_2 N(g)
\end{aligned} \tag{57}
$$

(u_m and r_m, $r_m \leqslant m^{-1}$, $(m = n_1, n_2, ...)$ are properly chosen).

Now we recall that [17, Lemma 3.1, p. 315]

$$M(g) = N(g).$$

Therefore

$$C_1 M(g) \leqslant C_2 m\omega\left(g; \frac{1}{m}\right) \leqslant C_2 M(g), \ (m = n_1, n_2, ...) \tag{58}$$

and from (54), (57), and (58)

$$\limsup_{n \to +\infty} \frac{\|L_n(g)-g\|_{[0,\pi]}}{\omega(g; 1/n)} \sim 1.$$

Finally from (56) and (55) we get (15).

Now we prove (16).

We quote an observation from [14] (see also [17, Proof of Theorem 2.2, p. 316]), namely

$$
\begin{aligned}
M(g; \theta) &= 0, \qquad \text{for each } \theta \in [0, \pi], \text{ iff } g = \text{ constant} \\
M(g) &< \infty, \qquad \text{iff } g \in \text{Lip } 1.
\end{aligned} \tag{59}
$$

Then if $g = \text{constant}$, then $L_n(g) - g = 0 = o(1/n)$; if $\|L_n(g)-g\|_{[0,\pi]} = o(1/n)$, then by (54) $M(g) = 0$ and by (59) $g = \text{constant}$. Further $g = \text{constant}$ iff $f = \text{constant}$, whence we get (16).

Finally we prove (17). If $\omega^{\varphi}(f; t) \leqslant Ct$, then by (13) it follows that

$$\|N_n(f)-f\| = O\left(\frac{1}{n}\right).$$

On the other hand, if $\|N_n(f)-f\| = \|L_n(g)-g\|_{0,\pi]} = O(1/n)$, then by (54) $M(g) < +\infty$. Then by (59) $g \in \text{Lip1}$ and by (55) we have (17). ∎

Now we prove Theorem 2.3.

Proof. From (46) if $s=2$, working as in [2, p. 83] by (55), we deduce (21) and (22).

Finally from the proofs of Lemmas 3.2 and 3.3 we get for $s=2$

$$\|\varphi N_n'(f)\| \leqslant Cn \, \|f\|, \qquad f \in C([-1,1]),$$

$$\|\varphi N_n'(f)\| \leqslant C \log n \, \|f'\varphi\|, \qquad f'\varphi \in C([-1,1]),$$

and working as in [7, Corollary 1, p. 81] by (55) we deduce (23). ∎

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